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GEOMETRICAL ORIGIN OF INTERNAL SYMMETRIES*

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In this note we wish to outline a group-theoretical method of obtaining internal symmetries of elementary particles from the geometry of space-time. The basis of our theory is the group T[1] having as the generators of its Lie algebra $\mathcal{T}$ the identity, momentum, and position operators I, P_μ, and X_μ, $\mu = 0, 1, 2, 3$, satisfying [2]

$$[P_\mu, X_\nu] = ig_{\mu\nu}I, \quad (\hbar = c = 1),$$

$$[P_\mu, P_\nu] = [X_\mu, X_\nu] = [P_\mu, I] = [X_\mu, I] = 0 ,$$

$$(g_{\mu\nu}) = \mathrm{diag}\,(1, -1, -1, -1) \quad .$$

These equations represent the simplest covariant generalization of the non-relativistic canonical commutation relations $[p_i, q_j] = -i\delta_{ij}$. Irreducible unitary representations of T are labeled by the eigenvalue σ (restricted to be > 0 for physical reasons) of I; for the maximal abelian subalgebra of $\mathcal{T}$ we choose the four P_μ.[3] The basic states are thus $|\sigma p\rangle$, to be physically interpreted as the momentum eigenstates of matter in which all spin, isospin, etc., dependence is "washed out" or ignored. We aim to show how these additional quantum numbers can be regenerated by starting with the basic states only.

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The allowed operations on the states are the usual ones in quantum mechanics: superposition, or formation of linear combinations of states with complex coefficients, and composition, or formation of tensor product states. Every "physical" state is to be obtained by a repeated application of these two operations to our basic states. The new states so obtained, if they are chosen to be the eigenstates of momentum, may again be regarded as the basic states if one suppresses their newly generated quantum numbers.

By passing to the enveloping algebra of $\mathcal{T}$, we may introduce the angular momentum operators $M_{\mu\nu} = X_{[\mu}P_{\nu]}/I = (X_\mu P_\nu - X_\nu P_\mu)/I = L_{\mu\nu}$. One finds the usual commutation relations between the operators P_μ, X_μ, and $M_{\mu\nu}$, namely, $\left[M_{\mu\nu}, P_\rho\right] = iP_{[\mu}g_{\nu]\rho}$, $\left[M_{\mu\nu}, M_{\rho\sigma}\right] = iM_{[\mu[\sigma}g_{\rho]\nu]}$, etc. Thus $M_{\mu\nu}$, if adjoined to $\mathcal{T}$, yields a new Lie algebra which is the Lie algebra of what we call the augmented Poincaré group P. [4] The total angular momentum M is equal to the orbital part L; this is true only for this simple case of the basic representations of T.

To illustrate the state-building process to the lowest order of complexity, let us consider the tensor product states $|\sigma_1 p_1\rangle \otimes |\sigma_2 p_2\rangle$ diagonalizing the operators $I(i)$ and $P_\mu(i)$, $i = 1, 2$, of the Lie algebra $\mathcal{T}(1) \oplus \mathcal{T}(2)$. Define the external Lie algebra $\mathcal{T}_{ext}$ (isomorphic to $\mathcal{T}$) by exhibiting its basis elements: $I = I(1) + I(2)$, $P_\mu = P_\mu(1) + P_\mu(2)$, $X_\mu = X_\mu(1) + X_\mu(2)$. The total angular momentum of this "two-particle system" is by definition just the sum of the individual angular momenta:

$$M_{\mu\nu} = M_{\mu\nu}(1) + M_{\mu\nu}(2) = L_{\mu\nu} + S_{\mu\nu} \quad , \tag{1}$$

where

$$L_{\mu\nu} = X_{[\mu} P_{\nu]}/I \quad ,$$

$$S_{\mu\nu} = \bar{X}_{[\mu} \bar{P}_{\nu]}/\bar{I} \quad ,$$

$$\bar{I} = I(1)\, I(2) > 0 \quad ,$$

$$\bar{P}_{\mu} = [P_{\mu}(1)\, I(2) - P_{\mu}(2)\, I(1)]\, I^{-\frac{1}{2}} \quad ,$$

and similarly for $\bar{X}_{\mu}$.[5] The operators $\bar{I}$, $\bar{P}_{\mu}$, and $\bar{X}_{\mu}$ form the internal Lie algebra $\mathcal{T}_{int}$ isomorphic and orthogonal to $\mathcal{T}_{ext}$, i.e., $[A, B] = 0$ for all A in $\mathcal{T}_{ext}$, B in $\mathcal{T}_{int}$. We may now diagonalize I, P_{μ}, $\bar{I}$, and $\bar{P}_{\mu}$ and so obtain the states $|\sigma\, p;\ \bar{\sigma}\, \bar{p}\rangle$ characterized by external and internal quantum numbers (relative to the group T). Equation (1) shows that we no longer have $M_{\mu\nu} = L_{\mu\nu}$. The spin part $S_{\mu\nu}$ of the total angular momentum arises as a consequence of the existence of the internal operators $\bar{P}_{\mu}$ and $\bar{X}_{\mu}$. The (intrinsic) spin tensor operator $S_{\mu\nu}$ is related to the usual particle spin through the equations $W^2 = -m^2\, s(s+1)$ and $W_{\mu} = \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma} S^{\nu\rho} P^{\sigma}$.[6] $S_{\mu\nu}$ obviously commutes with I, P_{μ}, and X_{μ} and hence with $L_{\mu\nu}$, as it should. Instead of the states $|\sigma\, p;\ \bar{\sigma}\, \bar{p}\rangle$, one could, e.g., take $|\sigma\, p\, s\, s_3;\ \bar{\sigma}\, f\, \kappa\rangle$, diagonalizing I, P_{μ}, $\underset{\sim}{S}^2$, S_3, $\bar{I}$, $\underset{\sim}{T}^2 - \underset{\sim}{S}^2$, and a certain combination of the operators $\bar{P}^2$ and $\bar{X}^2$. The various states so obtained are related to each other by the Clebsch-Gordan coefficients appropriate to T or P. In any case, it should now be clear how "complex" states may be generated from the basic ones. If one reduces an n-fold tensor product of the basic states, one obtains the states $|\sigma\, p;\ \bar{\sigma}_1 \bar{p}_1 \cdots \bar{\sigma}_{n-1} \bar{p}_{n-1}\rangle$ corresponding to one external and n-1 internal Lie algebras, all mutually orthogonal and isomorphic to $\mathcal{T}$.[7]

We proceed now to show how to obtain internal symmetry groups of particles with their traditionally "non-geometric" quantum numbers such as isospin, hypercharge, etc. Consider the above reduced n-fold tensor product states. The total angular momentum $M_{\mu\nu} = \sum_{k=1}^{n} M_{\mu\nu}(k)$ may again be decomposed into its orbital and spin parts. The spin tensor $S_{\mu\nu}$ now receives contributions from the n-1 internal Lie algebras; it is unique although its decomposition $S_{\mu\nu} = \sum_{k=1}^{n} \overline{X}_{[\mu}(k)\overline{P}_{\nu]}(k)/\overline{I}(k)$ depends on the choice of internal Lie algebras. If V_μ is any of the internal four-vector operators, then it obviously commutes with I, P_μ, and X_μ but not with $M_{\mu\nu}$: $[M_{\mu\nu}, V_\rho] = [S_{\mu\nu}, V_\rho] = iV_{[\mu}g_{\nu]\rho}$. The only combinations of internal operators commuting with the generators of the external Lie algebra of the augmented Poincaré group (and in particular with the elements of the Poincaré group itself) are the I's and the Lorentz scalars or invariants formed from the 2(n-1) internal four-vectors. To construct all the invariants it is useful to introduce the combinations

$$A_\mu^{\pm}(k) = [\ell_o\ \overline{P}_\mu(k) \pm i\ \overline{X}_\mu(k)/\ell_o]\,[2\overline{I}(k)]^{-\frac{1}{2}}\ ,\ k = 1, 2, \ldots, n-1,$$

where ℓ_o (the fundamental length!) has the dimensions of length or inverse mass and serves to make the A's dimensionless. Define

$$\xi_j^i = A_\mu^+(i)\ A^{-\mu}(j) = (\xi_i^j)^* \quad ,$$

$$\xi^{ij} = A_\mu^+(i)\ A^{+\mu}(j) = \xi^{ji} = (\xi_{ij})^* \quad .$$

By convention the upper and lower indices respectively correspond to the A^+'s and A^-'s; thus, e.g., $\xi_{ij} = A_\mu^-(i)\ A^{-\mu}(j)$. We find the following commutation relations:

$$[\xi_j^i, \xi_\ell^k] = \delta_\ell^i\ \xi_j^k - \delta_j^k\ \xi_\ell^i \quad ,$$

$$[\xi^{ij}, \xi^{k\ell}] = [\xi_{ij}, \xi_{k\ell}] = 0 \quad ,$$

$$[\xi^{ij}, \xi_{k\ell}] = \delta^i_k \xi^j_\ell + \delta^j_k \xi^i_\ell + \delta^i_\ell \xi^j_k + \delta^j_\ell \xi^i_k \;,$$
$$[\xi^{ij}, \xi^k_\ell] = \delta^i_\ell \xi^{jk} + \delta^j_\ell \xi^{ik} \,.$$

The first equation in conjunction with $(\xi^i_j)^* = \xi^j_i$ shows that the ξ^i_j are the generators of the unitary group $U_n \cong U_1 \times SU_n$. The remaining equations reveal that U_n is a subgroup of a larger internal symmetry group which is easily identified as the symplectic group C_n [8]. No higher internal symmetry group exists for this case of n-fold tensor product states because the only other possible internal invariants are of the form $\epsilon^{\mu\nu\rho\sigma} A^{\pm}_\mu(i)\, A^{\pm}_\nu(j)\, A^{\pm}_\rho(k)\, A^{\pm}_\sigma(\ell)$, and they fail to form a finite-dimensional Lie algebra. For fixed n, one may label the states by the quantum numbers associated with U_1 and the sequence of compact groups $SU_2 \subset SU_3 \subset \cdots \subset SU_n$.[9] Which subgroup SU_2 of SU_n should be identified with the physical isospin group can only be decided once a definite choice of internal Lie algebras is made and dynamical questions investigated. The fact that the SU_k are <u>subgroups</u> of C_n means that symmetry schemes based on any of them cannot be exact but are necessarily broken in a definite manner.

Further development of our theory, including its detailed physical interpretation and predictions; dynamical calculations; relations between internal and external quantum numbers, etc., will be presented in a comprehensive paper now being prepared.

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FOOTNOTES

1) The motivation for introducing T and the detailed discussion of it will be given elsewhere (J. S. Zmuidzinas, to be published). We merely quote the following 6 x 6 matrix realization of this group: T is the set of all matrices of the form

$$\begin{bmatrix} 1 & \tilde{v} & \alpha \\ 0 & I_4 & a \\ 0 & 0 & 1 \end{bmatrix} ,$$

where $\tilde{v} = (v^0, -v^1, -v^2, -v^3)$, α real, $I_4 = 4 \times 4$ unit matrix, and $a = \mathrm{col}(a^0, a^1, a^2, a^3)$, under the usual matrix multiplication.

2) The physical interpretation of these operators is the following. The P_μ are the usual momentum operators or generators of space-time translations. The position operators X_μ generate translations in the momentum space; they are in a sense the generators of dynamics, as one can easily convince himself if he thinks of their effect in mixing masses. The identity operator I is the generator of phases of quantum mechanical states [cf. the case of the Galilei group: Jean-Marc Levy-Leblond, Jour. Math. Phys. 4, 776 (1963)].

3) This choice is obviously neither unique nor necessary; however, it is convenient for physical interpretation. Another convenient choice is that of the four X_μ. In fact, the whole theory is invariant under $P_\mu \leftrightarrow X_\mu$, $I \longleftrightarrow -I$.

4) The group of invariant automorphisms of the Lie algebra (with the identity element I) of the restricted Poincaré group $P_+^\uparrow$ is very

closely related to the augmented Poincaré group P whose group law is

$$(\alpha', v', a', \ell')(\alpha, v, a, \ell)$$
$$= (\alpha' + \alpha + v' \cdot \ell' a, v' + \ell' v, a' + \ell' a, \ell'\ell);$$

T and $P_+^\uparrow$ are of course subgroups of P. We believe that the augmented Poincaré group is of fundamental significance in elementary particle physics.

5) All barred operators will henceforth be identified as internal.

6) The intrinsic spin operators $S_{\mu\nu} = (\underset{\sim}{S}, \underset{\sim}{T})$ form a Lie algebra which is that of the (proper homogeneous) Lorentz group; its irreducible unitary representations are characterized by the values of its two invariants $-\frac{1}{2} S_{\mu\nu}S^{\mu\nu} = f = 1 + \nu^2 - k^2$ and $\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma} = g = 2k\nu$, where ν takes a continuous range of values and k = 0, 1/2, 1, 3/2,... The eigenvalues of $\underset{\sim}{S}^2 = s(s + 1)$, s = k, k + 1, ..., represent the spin of a particle defined in its rest frame. The occurrence of half-integral spins should not be mysterious if one remembers that $S_{\mu\nu}$ is defined in terms of internal variables and that there is no reason to insist on the single-valuedness of any function of these unobservable variables.

7) The reduction process is clearly not unique: there are many possible sets of n-1 mutually orthogonal internal Lie algebras. The interesting possibility of choosing hierarchies of internal Lie algebras with progressively diminishing commutators $[\bar{P}, \bar{X}]$ will be discussed elsewhere. The existence of such hierarchies allows us to approximate states by neglecting higher order internal quantum numbers and thus a posteriori justifies our initial consideration of the basic states $|\sigma\, p\rangle$.

8) It is important to note that the group generated by the ξ's, although isomorphic to the complex group C_n, is not compact. This is most easily seen by examining C_1 which is contained in every C_n: the hermitian operators $K_1 = (\xi^{11} + \xi_{11})/4$, $K_2 = i(\xi^{11} - \xi_{11})/4$, and $K_3 = \xi_1^1/2$ generate the Lie algebra of the non-compact 3-dimensional Lorentz group.

9) A discussion of such hierarchy of unitary groups has recently been given by Neville [D. E. Neville, Phys. Rev. Letters 13, 118 (1964)]. See also C. R. Hagen and A. J. Macfarlane, Phys. Rev. 135, B434 (1964); and F. Gürsey and L. A. Radicati, Phys. Rev. Letters 13, 173 (1964).